Topology and its Applications 39 (1991) 245-259
North-Holland



# Some cardinal function relationships between $C_P(X)$ and finite powers of a space $X$

Basil Coutant

*Department of Mathematics, Ohio University, Athens, OH 45701, USA*

Received 5 June 1989
Revised 19 February 1990

*Abstract*

Coutant, B., Some cardinal function relationships between $C_P(X)$ and finite powers of a space $X$, Topology and its Applications 39 (1991) 245-259.

New bounds for the spread and hereditary density of the finite powers of the domain space $X$ are obtained in terms of $C_P(X)$, the space of continuous real-valued functions with the topology of pointwise convergence. Sharp bounds are found for spaces $X$ that include: spaces with a $G_\delta$-diagonal, topological vector spaces, and Lindelöf homogeneous spaces. Corollaries of these will give bounds on the spread and hereditary Lindelöf degree of $C_P(X)^2$.

*Keywords*: Space of continuous real-valued functions, hereditary Lindelöf degree, hereditary density, spread.

*AMS (MOS) Subj. Class.*: 54C30, 54C35, 54A25.

## 1. Introduction

In 1980/81 Zenor [9] and Veličko [8] independently proved the following cardinal function relationships between a Tychonoff space $X$ and its continuous, real-valued functions with the topology of pointwise convergence, $C_P(X)$.

(1) $\sup\{\mathrm{hL}(X^n)\colon n\in\mathbb{N}\}=\sup\{\mathrm{hd}[C_P(X)^n]\colon n\in\mathbb{N}\}$.

(2) $\sup\{\mathrm{hd}(X^n)\colon n\in\mathbb{N}\}=\sup\{\mathrm{hL}[C_P(X)^n]\colon n\in\mathbb{N}\}$.

(3) $\sup\{s(X^n)\colon n\in\mathbb{N}\}=\sup\{s[C_P(X)^n]\colon n\in\mathbb{N}\}$.

Veličko was able to improve the first of these by showing that $\mathrm{hd}[C_P(X)]=\mathrm{hd}[C_P(X)^n]$ for any positive integer $n$. He naturally asked whether this was true for the other two cardinal functions. It has been noted that the spread and hereditary Lindelöf degree of $C_P(X)^{n+2}$ is the same for all $n\in\mathbb{N}$ (see Corollary 1.5). So the question remains only for $C_P(X)$ and its square. These questions were recently asked again in Arhangel'skii's survey paper [1] on the theory of $C_P(X)$.

In this paper it is shown that $s[C_P(X)^2] \leqslant s[C_P(X)]^+$ (this inequality will imply the respective inequality for the hereditary Lindelöf degree). In addition, a variety of conditions are given for which the spreads of $C_P(X)$ and $C_P(X)^2$ must be equal.

We start by considering relationships between subsets $A \subseteq X$ and collections $\mathscr{F} \subseteq C_P(X)$, where $\mathscr{F}$ separates the points of $A$ from closed sets in $A$ ($\forall x \in X$ and $F \subseteq A$, $(x \notin \bar{F}) \Rightarrow (\exists f \in \mathscr{F}, f(x) \notin \overline{f(F)})$). Note that this is the definition used by most authors. However, some authors require that $f(x) = 0$ and $f(F) \subseteq [1, \infty)$. Lemma 1.1 shows that for the purposes of this paper the stronger condition may be assumed without any loss of generality. Finally, all spaces are assumed to be Tychonoff.

**Lemma 1.1.** *Suppose $\mathscr{F} \subseteq C_P(X)$ separates the points of $A \subseteq X$ from the closed sets in $A$. Then there is an $\mathscr{F}^* \subseteq C_P(X)$ such that for each point $x \in A$ and set $F \subseteq A$ $(x \notin \bar{F})$ there is an $f \in \mathscr{F}^*$ with $f(F) \subseteq [1, \infty)$ and $f(x) = 0$. Further for each cardinal function $\phi$ in $\{d, \mathrm{hd}, L, \mathrm{hL}, s\}$, $\phi(\mathscr{F}^*) \leqslant \phi(\mathscr{F})$.*

**Proof.** We define the new collection to be

$$\mathscr{F}^* = \{f_{n,q}(x) : f \in \mathscr{F},\ n \in \mathbb{N}, \text{ and } q \in \mathbb{Q}\}$$

where $f_{n,q}(x) = \max\{n|f(x) - q| - \frac{1}{2}, 0\}$. Then $\mathscr{F}^*$ is a continuous image of $\mathscr{F} \times \mathbb{N} \times \mathbb{Q}$. Hence for each cardinal function $\phi$ in $\{d, \mathrm{hd}, L, \mathrm{hL}, s\}$, $\phi(\mathscr{F}^*) \leqslant \phi(\mathscr{F})$. Further it will have the separation properties needed by selecting appropriate $n$, $q$ and $f$. □

**Proposition 1.2.** *If $A \subseteq X$ and $\mathscr{F} \subseteq C_P(X)$ separates the points in $A$ from closed sets in $A$, then*

(a) $\mathrm{hd}(A) \leqslant \mathrm{hL}(\mathscr{F})$;

(b) $\mathrm{hL}(A) \leqslant \mathrm{hd}(\mathscr{F})$; *and*

(c) $s(A) \leqslant s(\mathscr{F})$.

**Proof.** For proofs see [9] or [8]. □

The idea behind most of the theorems will be to construct collections $\mathscr{G} = \{\langle A_\alpha, \mathscr{F}_\alpha \rangle : \alpha \in \omega\}$ such that $\forall \alpha \in \omega$:

(i) $A_\alpha \subseteq X$ and $\mathscr{F}_\alpha \subseteq C_P(X)$;

(ii) $\mathscr{F}_\alpha$ separates the points of $A_\alpha$ from closed sets;

(iii) we have bounds on $\phi(\mathscr{F}_\alpha)$ for each of the cardinal functions $\phi$ in Proposition 1.2; and

(iv) $X = \bigcup\{A_\alpha : \alpha \in \omega\}$.

**Theorem 1.3.** (a) *There is a continuous image of $\sum_{n \in \omega} (X \times \mathbb{Q})^n$ in $C_P(C_P(X))$ that separates the points in $C_P(X)$ from closed sets in $C_P(X)$.*

(b) *There is a pair of continuous images of $C_P(X)$ in $C_P(X^2)$ that separates the points in $X^2$ from closed sets in $X^2$.*

(c) *For each $n \in \mathbb{N}$ there is a finite collection of continuous images of $C_P(X)^2$ in $C_P(X^n)$ that separates the points in $X^n$ from closed sets in $X^n$.*

**Proof.** (a) First we define $i_x : C_P(X) \to \mathbb{R}$ by $i_x(f) = f(x)$, $\forall x \in X$. Then the collection

$$\left\{ \sum_{\langle x,q\rangle \in J} |i_x - q| : J \in [X \times \mathbb{Q}]^{<\omega} \right\}$$

will work.

For (b) and (c) see the proof and the comments at the end of the proof of Lemma 3.1. □

**Definition 1.4.** For a cardinal function $\phi$ we define $\phi^*$ by

$$\phi^*(X) = \sup\{\phi(X^n) : n \in \omega\}.$$

By theorems of Zenor (see [9])

(1) $s^*(X) = \min\{\mathrm{hL}^*(X), \mathrm{hd}^*(X)\}$ and

(2) $\phi^*(X) = \phi(X^\omega)$ for $\phi \in \{s, \mathrm{hd}, \mathrm{hL}\}$ (in general we have $\phi(X^\kappa) = \phi^*(X) \cdot \kappa$).

**Corollary 1.5** (Veličko). *For any space $X$*

(a) $\mathrm{hL}(X^2) \leq \mathrm{hd}[C_P(X)] \leq \mathrm{hd}[C_P(X)^2] = \mathrm{hL}^*(X)$;

(b) $\mathrm{hd}(X^2) \leq \mathrm{hL}[C_P(X)] \leq \mathrm{hL}[C_P(X)^2] = \mathrm{hd}^*(X)$; *and*

(c) $s(X^2) \leq s[C_P(X)] \leq s[C_P(X)^2] = s^*(X)$.

**Remarks 1.6.** (1) Since we are only considering Tychonoff spaces, $s(X) \leq \mathrm{hL}(X)$, $\mathrm{hd}(X) \leq \mathrm{nw}(X) = \mathrm{nw}[C_P(X)] \leq 2^{s(X)}$. Hence all of the above cardinals must be between $s(X)$ and $2^{s(X)}$. (Showing that $\mathrm{nw}(X) = \mathrm{nw}[C_P(X)]$ is an easy exercise using the fact that $X$ can be embedded in $C_P C_P(X)$. The other inequalities are in [5].)

(2) $C_P(X)^\omega$ is homeomorphic to $C_P(X \times \omega)$. Thus we easily get that $\phi^*[C_P(X)] = \phi[C_P(X)^\omega]$, since $\phi^*(X) = \phi^*(X \times \omega)$ for $\phi$ in $\{\mathrm{hd}, \mathrm{hL}, s\}$.

We conclude Section 1 by noting two cases for which Corollary 1.5 can be improved.

**Proposition 1.7.** *If a compact space $X$ contains a copy of $X \times 2$, then there is a continuous map from $C_P(X)$ onto $C_P(X)^2$. Hence $\forall n \in \mathbb{N}$, $\phi[C_P(X)^n] = \phi[C_P(X)]$.*

**Proof.** Use the restriction mapping. By the Urysohn extension theorem the range is $C_P(X \times 2)$, which is homeomorphic to the square. □

As we will see in Theorem 4.6, even if $X$ is not compact, it is still true that if $X$ contains a copy of $X \times 2$, then for $n \in \mathbb{N}$, $\phi[C_P(X)^n] = \phi[C_P(X)]$.

**Theorem 1.8** (D.S. Pavlovskii). *If $X$ is a 0-dimensional space, then*

(a) $\mathrm{hd}^*(X) = \mathrm{hL}[C_P(X)]$;

(b) $\mathrm{hL}^*(X) = \mathrm{hd}[C_P(X)]$; *and*

(c) $s^*(X) = s[C_P(X)]$.

**Proof.** Modify the proof of Lemma 2.1/3.1 below using characteristic functions. $\square$

**Remark 1.9.** It suffices for the clopen sets in Theorem 1.8 to form a pseudo-base for $X$.

## 2.

Next we have two versions of a lemma (Lemmas 2.1 and 3.1). The first will give us short proofs of: Veličko's theorem ($\mathrm{hd}[C_P(X)] = \mathrm{hd}[C_P(X)^2]$), an extension of Asanov's theorem [3] that if $\mathrm{hL}[C_P(X)] = \Delta(X) = \omega$ then $\mathrm{hd}^*(X) = \omega$ to the dual theorems ($\mathrm{hL}[C_P(X)] = \mathrm{hL}[C_P(X)^2]$ and $s[C_P(X)] = s[C_P(X)^2]$) for spaces with a $G_\delta$-diagonal, the dual theorems for Corson and Eberlein compact spaces, the dual theorems for the countable case under PFA, and that extent of $X^n$, $e(X^n)$, is bounded by $s[C_P(X)]$. The second version will be used to show that $s(X^n)$ is at most $s[C_P(X)]^+$. Both versions essentially state that for sets in $X^n$ far away from the diagonals,

$$\Delta_{i,j} = \{\langle x_k : k \in n\rangle \in X^n : x_i = x_j\},$$

we can get bounds on the cardinal functions hd, hL, and $s$ using collections which separate points from closed sets.

**Lemma 2.1.** *Let $n \in \mathbb{N}$. If $A \subseteq X^n$ is such that $\forall i, j < n$, $\bar{A} \cap \Delta_{i,j} = \emptyset$, then there is an $\mathcal{F} \subseteq C_P(X^n)$ which is a continuous image of $C_P(X)$ that separates the points in $A$ from closed sets in $A$.*

**Proof.** See Lemma 3.1. $\square$

**Theorem 2.2.** *For $n \in \mathbb{N}$, $e(X^n) \leqslant s[C_P(X)]$.*

**Proof.** We use induction on $n$. First for $n = 1$ we have $e(X) \leqslant s(X) \leqslant s[C_P(X)]$ by Corollary 1.5. So assume that the theorem is true for $n \in \mathbb{N}$ and let $E$ be a closed, discrete subset in $X^{n+1}$. Then we have

(1) $|E \cap (\bigcup \Delta_{i,j})| \leqslant s[C_P(X)]$ by the induction hypothesis and

(2) $|E \setminus (\bigcup \Delta_{i,j})| \leqslant s[C_P(X)]$ by Lemma 2.1.

So $|E| \leqslant s[C_P(X)]$ as desired. $\square$

**Theorem 2.3.** *Let $X$ be a Tychonoff space. Then $\forall n \in \mathbb{N}$*

(a) $\mathrm{hd}(X^n) \leqslant \mathrm{hL}[C_P(X)] \cdot \Delta(X)$;

(b) $\mathrm{hL}(X^n) \leqslant \mathrm{hd}[C_P(X)] \cdot \Delta(X)$; *and*

(c) $s(X^n) \leqslant s[C_P(X)] \cdot \Delta(X)$.

(*In particular the conjecture holds for spaces with a $G_\delta$-diagonal.*)

**Proof.** For each case the proof may be completed by showing that $X^n \setminus \bigcup \Delta_{i,j}$ is the union of a collection of $\{A_\alpha : \alpha \in \Delta(X)\}$ where each $A_\alpha$ meets the conditions of

Lemma 2.1. This may be done by showing that $\bigcup \Delta_{i,j}$ is the intersection of $\Delta(X)$ many open sets. This however holds by the definitions of the $\Delta_{i,j}$'s and $\Delta(X)$, and the fact that the finite union of $G_\kappa$'s is still a $G_\kappa$. □

**Corollary 2.4** (Veličko). *For any Tychonoff space X,* $\mathrm{hL}^*(X) = \mathrm{hd}[C_\mathrm{P}(X)]$.

**Proof.** For any regular space we have $\Delta(X) \leq \mathrm{hL}(X^2)$. □

In fact if $\mathscr{F} \subseteq C_\mathrm{P}(X)$ separates the points of $X$, then $\Delta(X) \leq \mathrm{iw}(X) \leq d(\mathscr{F}) = d[C_\mathrm{P}(X)]$ (see [7]). Hence for any space $X$, $\Delta(X) \leq d[C_\mathrm{P}(X)] \leq \mathrm{hd}[C_\mathrm{P}(X)]$.

**Proposition 2.5.** *If $X$ is weakly $\delta\theta$-refinable, then* $L(X) \leq s(X)$.

**Proof.** Let $\mathscr{U}$ be an open cover of $X$ and $\mathscr{V} = \langle \mathscr{V}_n : n \in \mathbb{N} \rangle$ a weak $\delta\theta$-refinement of $\mathscr{U}$. For each $n \in \mathbb{N}$ define

$$L_n = \{x \in \bigcup \mathscr{V}_n : \mathrm{ord}(x, \mathscr{V}_n) \leq \omega\}.$$

Next choose discrete subsets $D_n$ of $L_n$ to be maximal with respect to the property that $\{x, y\} \in [D_n]^2$ implies $x \notin \mathrm{st}(y, \mathscr{V}_n)$.

Then $|D_n| \leq s(X)$ and $L_n \subseteq \bigcup \{\mathrm{st}(x, \mathscr{V}_n) : x \in D_n\}$ for each $n \in \mathbb{N}$. Hence $\mathscr{U}$ has a subcover of cardinality $\leq s(X)$, since the $L_n$'s cover $X$ and each $x \in D_n$ is in at most countably many members of $\mathscr{V}_n$. □

**Corollary 2.6.** *If $X^2 \backslash \Delta$ is weakly $\delta\theta$-refinable, then $\Delta(X) \leq s(X^2)$. Consequently* $s^*(X) = s[C_\mathrm{P}(X)]$ *and* $\mathrm{hd}^*(X) = \mathrm{hL}[C_\mathrm{P}(X)]$.

**Theorem 2.7** (Gruenhage [4]). *A compact space X is Corson (Eberlein) compact iff $X^2 \backslash \Delta$ is metalindelöf ($\sigma$-metacompact).*

**Corollary 2.8.** *For each of the above compact cases we have* $\mathrm{hd}^*(X) = \mathrm{hL}[C_\mathrm{P}(X)]$ *and* $s^*(X) = s[C_\mathrm{P}(X)]$.

We conclude this section with two consistency results for the countable case under PFA. The second one has been noted by Arhangel'skii [2] is a slightly different form.

**Lemma 2.9** (see Juhász's chapter of [6]). (PFA) *For a regular space X if $s(X) = \omega$, then X is hereditarily Lindelöf.*

**Theorem 2.10.** (PFA) *If $s(X^2) = \omega$ (and hence if $s[C_\mathrm{P}(X)] = \omega$), then* $\mathrm{hd}^*(X) = \mathrm{hL}[C_\mathrm{P}(X)]$ *and* $s^*(X) = s[C_\mathrm{P}(X)]$. *If $s[C_\mathrm{P}(X)] = \omega$, then* $\mathrm{hd}^*(X) = \mathrm{hL}[C_\mathrm{P}(X)] = s^*(X) = s[C_\mathrm{P}(X)] = \mathrm{hL}^*(X) = \mathrm{hd}[C_\mathrm{P}(X)] = \omega$.

**Proof.** By Lemma 2.9, $\Delta(X) \leq s(X^2) = \omega$. Hence by Theorem 2.3 we get the first two equalities.

If $s[C_P(X)]=\omega$, then $\Delta(X)=\omega$ by Corollary 1.5 and Lemma 2.9. Hence by Theorem 2.3 $s[C_P(X)^2]=s^*(X)=\omega$. But then Corollary 1.5 and Lemma 2.9 imply the rest of the equalities. $\square$

## 3.

We now improve Lemma 2.1 so that it may be applied more readily to an induction on the finite powers of $X$.

**Lemma 3.1** (The alternate version of Lemma 2.1). *Suppose $A\subseteq X^n$ is such that*

(1) $\forall 0<i<j<n$, $\bar{A}\cap\Delta_{i,j}=\emptyset$ *and*

(2) $\forall i<j<n$, $A\cap\Delta_{i,j}=\emptyset$.

*Then there is a continuous image of $C_P(X)$ in $C_P(X^n)$ that separates the points of $A$ from closed sets in $A$.*

**Proof.** Define $\Psi: C_P(X)\to C_P(X^n)$ by

$$\Psi(f)(\langle x_i: i\in n\rangle)=\left(\sum_{i\in n\setminus 1}|f(x_i)-i|\right)+|f(x_0)+1|.$$

Then $\Psi$ is continuous with continuous images since $n$ is finite.

To see that this has the required separation property let $a=\langle a_i: i\in n\rangle\in A$ and $F\subseteq A$ with $a\notin\bar{F}$. Define $Z$ to be the finite set

$$Z=\{a_i: i\in n\}^n\cap\left(\bigcup\{\Delta_{i,j}: 0<i<j<n\}\cup\{a\}\right).$$

Note that by the hypothesis $Z$ does not meet the closure of $F$. Hence we may choose neighborhoods $V_i$ of $a_i$ such that

(1) $\{V_i: i\in n\}$ is pairwise disjoint and

(2) for each $\langle a_{i_k}: k\in n\rangle\in Z$ we have $(\prod V_{i_k})\cap F=\emptyset$.

Now choose $f\in C_P(X)$ such that

(3) $f(a_0)=-1$ and $f(a_i)=i$ for $i\in n\setminus 1$;

(4) $f(V_0)\subseteq[-1,0]$ and $f(V_i)\subseteq[0,i]$ for $i\in n\setminus 1$;

(5) $f(X\setminus\bigcup V_i)\subseteq\{0\}$.

To do this choose $g_0$ such that $g_0(a_0)=-1$ and $g_0(X\setminus V_0)=\{0\}$. Next for $i\in n\setminus 1$ choose $g_i$ such that $g_i(a_i)=i$ and $g_i(X\setminus V_i)=\{0\}$. Then let $f=\sum g_i$.

**Claim.** *If $y=\langle y_i: i\in n\rangle\in F$, then $\Psi(f)(y)\geqslant 1$.*

Assume not. Then for each $i\in n\setminus 1$ we have $|f(y_i)-i|<1$ and $|f(y_0)+1|<1$. So $y_0\in V_0$. And for each $i\in n\setminus 1$

$$y_i\in\bigcup\{V_j: i<j<n\}.$$

Let $i_k \in n$ be such that $y_k \in V_{i_k}$. (There is only one choice since the $V_i$'s are pairwise disjoint.) By noting that $\langle a_{i_k} : k \in n\rangle \in Z$ we get a contradiction to $(\prod V_{i_k}) \cap F = \emptyset$. □

The proof of Lemma 2.1 is easier since everything is symmetric. In addition the proofs of Lemmas 1.3(b) and (c) may be obtained with slight variations. For (b) we have only one diagonal to consider so the above lemma takes care of this. Use the set

$$\{\sum |f(x_i) - 1| : f \in C_P(X)\}.$$

For the points on the diagonal. For (c) we may use two independent functions. Choose one as in Lemma 3.1 and the one that would have been chosen if the coordinates had been reversed. Add the resulting functions in $C_P(X^n)$ to get the desired function.

Next are two technical results needed to prove the main theorem in this section, $s[C_P(X)^2] \leq s[C_P(X)]^+$. The unusual form (at least for a proof in ZFC) of this theorem originates in Theorem 3.3. The theorem is necessarily stated in a rather cumbersome form, however, the basic technique is quite simple.

**Lemma 3.2.** *Let $A \subseteq X \times Y$ and $\mathcal{V} = \{(X \times b_i) \cap A : i \in I\}$ be a pairwise disjoint collection of sets where each $b_i$ is open. Then $|\mathcal{V}| \leq s(Y)$.*

**Proof.** For each $i \in I$ choose $\langle x_i, y_i\rangle \in (X \times b_i) \cap A$. Then the set $\{y_i : i \in I\}$ is discrete in $Y$. Since if $y_j \in b_i$, for $i \neq j$, then $\langle x_j, y_j\rangle \in (X \times b_i) \cap A$. So

$$\langle x_j, y_j\rangle \in ((X \times b_i) \cap A) \cap ((X \times b_j) \cap A),$$

a contradiction. □

**Theorem 3.3.** *Let $A \subseteq X \times Y$ with $s(Y) \leq \lambda^+$. Further assume that each $a$ in $A$ is contained in an open set $X \times b_a$ such that $|X \times b_a \cap A| \leq \lambda$. Then $|A| \leq \lambda^+$.*

**Proof.** Inductively define sets $A_\alpha \subseteq A$ and $S_\alpha \subseteq A_\alpha$ for $\alpha \in \lambda^+$. Given $\{A_\beta : \beta \in \alpha\}$ and $\{S_\beta : \beta \in \alpha\}$ consider the set $A \setminus \bigcup_{\beta \in \alpha} A_\beta$. If $A \setminus \bigcup_{\beta\in\alpha} A_\beta$ is empty let $S_\alpha$ and $A_\alpha$ both be empty. Otherwise first choose $S_\alpha$ such that

(i) $\{(X \times b_a) \cap (A \setminus \bigcup_{\beta\in\alpha} A_\beta) : a \in S_\alpha\}$ is a maximal pairwise disjoint collection in $A \setminus \bigcup_{\beta\in\alpha} A_\beta$. Next let

(ii) $A_\alpha = \bigcup \{(X \times b_a) \cap (A \setminus \bigcup_{\beta\in\alpha} A_\beta) : a \in S_\alpha\}$.

Notice that if $a \in A \setminus \bigcup \{A_\beta : \beta \in \alpha + 1\}$ then $X \times b_a$ meets $A_\alpha$ since otherwise

$$\{(X \times b_s) \cap (A \setminus \textstyle\bigcup_{\beta\in\alpha} A_\beta) : s \in S_\alpha \cup \{a\}\}$$

would be a disjoint collection. Also notice that by construction the $A_\alpha$'s are pairwise disjoint.

Now we have $|S_\alpha| \leq \lambda^+$ for each $\alpha \in \lambda^+$ by Lemma 3.2 and hence $\bigcup \{A_\alpha : \alpha \in \lambda^+\}$ will also have cardinality at most $\lambda^+$.

**Claim.** $A = \bigcup A_\alpha$.

Assume not and let $a \in A \setminus \bigcup A_\alpha$. Then as noted above from first induction condition $(X \times b_a) \cap A_\alpha \neq \emptyset$ for each $\alpha \in \lambda^+$. But then $|X \times b_a \cap A| \geq \lambda^+$ since the $A_\alpha$'s are pairwise disjoint. □

**Theorem 3.4.** *For any* $T_{3\frac{1}{2}}$ *space* $X$ *and* $n \in \mathbb{N}$, $s(X^n) \leq s[C_P(X)]^+$.

**Proof.** Use induction. The cases $n = 1$ and 2 are done. So let $n \geq 3$ and consider a discrete set $Y \subseteq X^n$. Without loss of generality $Y \cap \Delta_{i,j} = \emptyset$ for each $i < j < n$. So for each $y = \langle y_k : k \in n \rangle \in Y$ choose neighborhoods $V_y^k$ of $y_k$ such that

$$\left( \prod_{k \in n} \overline{V_y^k} \right) \cap \left( \bigcup \Delta_{i,j} \right) = \emptyset.$$

Let $V_y^* = X \times (\prod_{k \in n \setminus 1} V_y^k)$. Now by the above and Lemma 3.1

$$|V_y^* \cap Y| \leq s[C_P(X)].$$

So we may apply Theorem 3.3 to get that $|Y| \leq s[C_P(X)]^+$. Therefore by induction we get the theorem to be true for all $n$. □

Note that at least consistently there are locally countable, uncountable regular spaces with countable spread. (The Kunen line is such an example.) Hence we cannot expect to get equality from the hypotheses that allowed us to apply Theorem 3.3.

**Theorem 3.5** (Veličko [8]). *For any Tychonoff space* $X$ *and* $\forall n \in \mathbb{N}$ $\mathrm{hd}(X^n) \leq \mathrm{hL}[C_P(X)] \cdot s(X^n)$.

**Proof.** (Veličko's) Again use induction with the cases 1 and 2 done. Let $\lambda = \mathrm{hL}[C_P(X)]$, $\kappa = s(X^n)$ and $A \subseteq X^n$. Without loss of generality $A \cap (\bigcup \Delta_{i,j}) = \emptyset$ for each $i$ and $j < n$. Now $\forall a \in A$ there is a neighborhood $V_a \subseteq A$ with $\bar{V}_a \cap (\bigcup \Delta_{i,j}) = \emptyset$ for each $i$ and $j < n$. Inductively choose $a_\alpha$'s such that

$$a_\alpha \in A \setminus \left( \bigcup \{ V_{a_\beta} : \beta \in \alpha \} \cup \overline{\{ a_\beta : \beta \in \alpha \}} \right).$$

Then the collection of $a_\alpha$'s will be discrete and hence there is at most $\kappa$ of them. Further by Lemma 3.1 each of the $V_{a_\alpha}$'s will have a dense set $D_\alpha$ with cardinality at most $\lambda$. So the set $D = \bigcup D_\alpha$ will be the required dense set (each $a_\alpha$ is in $V_{a_\alpha} \subseteq \bar{D}_\alpha$). □

**Corollary 3.6.** *For any space*

(i) $s[C_P(X)] \leq s[C_P(X)^\omega] \leq s[C_P(X)]^+$ *and*

(ii) $\mathrm{hL}[C_P(X)] \leq \mathrm{hL}[C_P(X)^\omega] \leq \mathrm{hL}[C_P(X)]^+$.

*Further, if* $s[C_P(X)] < \mathrm{hL}[C_P(X)]$, *then* $\mathrm{hL}[C_P(X)] = \mathrm{hL}[C_P(X)^\omega]$.

**Remarks 3.7.** (1) The inequality $\mathrm{hL}[C_P(X)^\omega] \leq \mathrm{hL}[C_P(X)]$ fails only if the corresponding inequality for the spread fails and in this case the hereditary Lindelöf degree and the spread of $C_P(X)$ are the same.

(2) If $s^*(X)$ is a limit cardinal then the equalities $s^*(X) = s[C_P(X)]$ and $\mathrm{hd}^*(X) = \mathrm{hL}[C_P(X)]$ must hold. So in any space for which these equalities do not hold we must have that

$$\sup\{s(X^n) : n \in \mathbb{N}\} = s(X^m) \quad \text{for some } m \in \mathbb{N} \ (m \geq 3).$$

## 4.

In this section we will show that for a wide variety of spaces the equalities $s^*(X) = s[C_P(X)]$ and $\mathrm{hd}^*(X) = \mathrm{hL}[C_P(X)]$ do hold. In all of these we will achieve our results by assuming the existence of continuous functions from $X$ (or $X^2$) to $X$.

**Theorem 4.1.** *Suppose there is a countable collection of continuous functions $\Psi_i : X^2 \to X$, $i \in I$, such that*

(a) $\forall x \in X, \forall i \in I, \Psi_i(\langle x, x\rangle) = x$ *and*

(b) $\forall \langle x, y\rangle \in X^2 \setminus \Delta, \exists i \in I$ *with* $\Psi_i(\langle x, y\rangle) \notin \{x, y\}$.

*Then* $\mathrm{hd}^*(X) = \mathrm{hL}[C_P(X)]$ *and* $s^*(X) = s[C_P(X)]$.

**Lemma 4.2.** *Suppose there is a countable collection of continuous functions $\Psi_i : X^2 \to X$, $i \in I$, as above. Then for each $n \in \mathbb{N}$ and function $L$ from $n^2$ to $I$ there is a continuous function $H_L : C_P(X) \to C_P(X^n)$ such that $\vec{H}_L C_P(X)$ separates points from closed sets in $Y_L$ where*

$$Y_L = \{\langle x_i : i \in n\rangle \in X^n : \forall i < j < n, \Psi_{L(i,j)}\langle x_i, y_j\rangle \notin \{x_k : k \in n\}\}.$$

Condition (a) in Theorem 4.1 forces the $Y_L$'s to be subsets of $X^n \setminus \bigcup \Delta_{i,j}$. For these sets we can very nearly separate points from closed sets. The problem occurs when one of the elements of the closed set is near a particular point of a diagonal. We will use the auxiliary functions in the hypothesis to check for just this type of point.

**Proof of Lemma 4.2.** Fix $n \in \mathbb{N}$ and the function $L \in {}^{n^2}I$. Define $H$ $(= H_L)$ by

$$H(f)\langle x_i : i \in n\rangle = \sum_{i \in n} |f(x_i) - (i+1)| + \sum_{i<j<n} |f(\Psi_{L(i,j)}\langle x_i, x_j\rangle + 1|.$$

Fix $y = \langle y_i : i \in n\rangle \in Y_L$ and a closed $F \subseteq X^n \setminus \{y\}$. Now choose open sets $U_i$ and $W_{i,j}$ such that

(a) $y \in \prod_{i \in n} U_i \subseteq X^n \setminus F$:

(b) $y_i \in U_i$, $i \in n$ and $\Psi_{L(i,j)}\langle y_i, y_j\rangle \in W_{i,j}$, $i < j < n$;

(c) the collection $\{U_i : i \in n\} \cup \{W_{i,j} : i < j < n\}$ is pairwise disjoint.
Note if $\Psi_{L(i,j)}\langle y_i, y_j\rangle = \Psi_{L(r,s)}\langle y_r, y_s\rangle$ we have $W_{i,j} = W_{r,s}$.

Also choose neighborhoods $V_i$ of $y_i$ such that

(d) $V_i \subseteq U_i$ and $V_i^2 \subseteq \bigcap_{i<j<n} \bar{\Psi}_{L(i,j)} U_i$. ($\Psi\langle x, x\rangle = x$.)

Next we may choose a suitable $f \in C_P(X)$ such that

(i) $f(y_i) = i+1$;

(ii) $f(V_i) \subseteq [0, i+1]$;

(iii) $f(\Psi_{L(i,j)}\langle y_i, y_j\rangle) = -1$;

(iv) $f(W_{i,j}) \subseteq [-1, 0]$;

(v) $f(X \setminus [\bigcup V_i \cup \bigcup W_{i,j}]) \subseteq \{0\}$.

This may be done by adding up Tychonoff functions such that (i)-(iv) are met separately (being careful not to use more than one such function for each distinct point). Then by (c) above the sum will have the desired properties.

**Claim.** *$G = H(f)$ separates the point $y$ from the closed set $F$.*

**Proof.** First note that $G(y) = 0$ by conditions (i) and (iii). Let $z = \langle z_i : i \in n\rangle \in X^n$ and suppose that $G(z) < \frac{1}{2}$. We will show that $z \in \prod V_i$ and hence is not in $F$.

Since $G(z) < \frac{1}{2}$ we have $|f(z_i) - (i+1)| < \frac{1}{2}$ for each $i$ in $n$. Consequently we have

$$z_i \in \bigcup_{i \leq j} V_j \quad \text{for each } i \in n.$$

If there is a $k \in n$ such that $z_k \notin V_k$, then by the pairwise disjointness of the $V$'s $\exists i < j < n$ and $k \in n$ with $\{z_i, z_j\} \subseteq V_k$, i.e.

$$\langle z_i, z_j\rangle \in V_k^2.$$

But this implies that $\Psi_{L(i,j)}\langle z_i, z_j\rangle \in U_k$ by (d). So by the pairwise disjointness in (c) and conditions (ii), (iv) and (v)

$$f(\Psi_{L(i,j)}\langle z_i, z_j\rangle) \geq 0.$$

Hence $G(z) \geq |f(\Psi_{L(i,j)}\langle z_i, z_j\rangle) + 1| \geq 1$ a contradiction. So we must have that $z_i \in V_i$ for each $i$. Hence we have $z \in \prod V_i$.

In order to finish the proof we need to show that $H$ is continuous with continuous images. To see this simply note that if $\langle x_i : i \in n\rangle$ is in $X^n$, then the collection

$$\{x_i : i \in n\} \cup \{\Psi_{L(i,j)}\langle x_i, x_j\rangle : i < j < n\}$$

is finite. So if $\mathcal{M}$ is a net in $C_P(X)$ that converges to $f$, then the values for points in the finite set converge to $f$ of that point. Consequently $H(g)\langle x_i : i \in n\rangle$ will converge to $H(f)\langle x_i : i \in n\rangle$. So $H$ is continuous. The images will be continuous since, for a fixed $f$, $H(f)$ is a finite combination of sums and compositions of continuous functions. □

**Lemma 4.3.** *Let $n \in \mathbb{N}\setminus\{1,2\}$. Then $X^n$ may be written as the countable union of sets that are either a $Y_L$ as above or a continuous image of $X^{n-1}$.*

**Proof.** For $i<j<n$, $s \in I$ and $k \in n\setminus\{i,j\}$ define $\Gamma^{i,j}_{k,s}: X^{n-1} \to X^n$ by $\Gamma^{i,j}_{k,s}\langle x_r: r \in n-1\rangle = \langle y_r: r \in n\rangle$ where

$$y_r = \begin{cases} x_r, & r<k, \\ \Psi_s\langle x_i, x_j\rangle, & r=k, \\ x_{r-1}, & r>k. \end{cases}$$

Then we will have that

$$X^n = \bigcup_{i<j<n} \Delta_{i,j} \cup \bigcup_{L \in {}^{n^2}I} Y_L \cup \bigcup_{\substack{i<j<n \\ k \in n\setminus\{i,j\} \\ s \in I}} \Gamma^{i,j}_{s,k}(X^{n-1}).$$

Let $\langle x_i: i \in n\rangle \in X^n \setminus \bigcup \Delta_{i,j}$. Then for each pair $i<j<n$ $\exists s_{i,j} \in I$ such that $\Psi_{s_{i,j}}\langle x_i, x_j\rangle \notin \{x_i, x_j\}$. If in addition we have that $\Psi_{s_{i,j}}\langle x_i, x_j\rangle \notin \{x_i: i \in n\}$ for each $i<j<n$ then the point is in $Y_L$. Where

$$L(i,j) = \begin{cases} s_{i,j}, & \text{if } i<j, \\ s_{0,1}, & \text{else.} \end{cases}$$

Otherwise we have that

$$\langle x_i: i \in n\rangle \in \Gamma^{i,j}_{s_{i,j},k}(X^{n-1}) \quad \text{for some } k \in n\setminus\{i,j\}. \qquad \square$$

The proof of Theorem 4.1 is then just a simple induction argument.

**Corollary 4.4.** *If $X$ is a vector space, then* $\mathrm{hd}^*(X) = \mathrm{hL}[C_P(X)]$ *and* $s^*(X) = s[C_P(X)]$. (*In particular if $X = C_P(Y)$ for any $Y$.*)

**Proof.** Define $\Psi: X^2 \to X$ by $\Psi(x,y) = \frac{1}{2}(x+y)$. $\Psi$ satisfies all the requirements of Theorem 4.1. $\square$

**Theorem 4.5.** *Suppose there is a countable collection of homeomorphic embeddings $h_i: X \to X$, $i \in I$, such that $\forall x \in X$, $\exists i \in I$ with $h_i(x) \neq x$. Then* $\mathrm{hd}^*(X) = \mathrm{hL}[C_P(X)]$ *and* $s^*(X) = s[C_P(X)]$.

The proof is similar to that of Theorem 4.1. We use induction on $\mathbb{N}$ by writing $X^n$ as the countable union of sets that are either

(a) a continuous image of a subset of $X^{n-1}$ or

(b) sets $Y_L$ with a continuous $H_L: C_P(X) \to C_P(X^n)$, such that $H_L[C_P(X)]$ separates the points of $Y_L$ from closed sets.

**Lemma 4.6.** *If $\{h_i : i \in I\}$ is a collection of continuous functions from $X$ to $X$, then for each $L \in {}^nI$ there is a continuous $H_L : C_P(X) \to C_P(X^n)$ that separates the points of*

$$Y_L = \{\langle x_i : i \in n\rangle \in X^n : \forall i, j \in n (h_{L(i)}(x_i) \notin \{x_k : k \in n\}) \\ \wedge i < j \Rightarrow (x_i \neq x_j \wedge h_{L(i)}(x_i) \neq h_{L(j)}(x_j) \\ \wedge h_{L(i)}(x_j) \notin \{h_{L(k)}(x_k) : k \in n \backslash \{j\}\})\}$$

*from closed sets.*

**Proof.** Fix $L \in {}^nI$. Define $H_L : C_P(X) \to C_P(X^n)$ by

$$H_L(f)\langle x_i : i \in n\rangle = \sum_{i \in n} |f(x_i) - (i+1)| + |f(h_{L(i)}(x_i)) + (n - i)|$$

**Claim 1.** *$H_L : C_P(X) \to C_P(X^n)$ is continuous.*

**Proof.** First we have that for fixed $f \in C_P(X)$, $H_L(f)$ is a continuous function from $X^n$ to $\mathbb{R}$ since it is a finite combination of compositions with continuous functions and sums.

Next $H_L$ is continuous. Let $\langle x_i : i \in n\rangle$ be a point in $X^n$. Then $\{x_i : i \in n\} \cup \{h_{L(i)}(x_i) : i \in n\}$ is a finite set in $X$. So if we have pointwise convergence in $C_P(X)$, then we have convergence at each of the points in this set.

**Claim 2.** *The image of $H_L$ separates the points of $Y_L$ from closed sets.*

**Proof.** Fix $y = \langle y_i : i \in n\rangle \in Y_L$ and a closed set $A \subseteq X^n \backslash \{y\}$. We need to find an $f \in C_P(X)$ such that $H_L(f)$ separates $y$ and $A$.

For $i \leq j < n$ choose open sets $U_i$, $V_i$, and $W_{i,j}$ such that

(i) $\forall i \in n$, $y_i \in V_i \subseteq U_i$;

(ii) $\forall i \leq j < n$, $h_{L(i)}(y_j) \in W_{i,j}$;

(iii) the collection $\{U_i : i \in n\} \cup \{W_{i,j} : i \leq j < n\}$ is pairwise disjoint (note if there is any repetition in the labeling of the points $\{y_i : i \in n\} \cup \{h_{L(i)}(y_j) : i \leq j < n\}$, then the same open neighborhoods are to be chosen with a corresponding repetition of indices);

(iv) $(\prod_{i \in n} U_i) \cap A = \emptyset$;

(v) $\forall j \in n$, $V_j \subseteq \bigcap_{i \leq j} h^{\leftarrow}_{L(i)} W_{i,j}$.

Next choose (or construct) a function $f \in C_P(X)$ such that

(a) $\forall i \in n, f(y_i) = i + 1$;

(b) $\forall i \in n, f(V_i) \subseteq [0, i+1]$;

(c) $\forall i \in n, f(h_{L(i)}(y_i)) = -n + i$;

(d) $\forall i \in n, f(W_{i,i}) \subseteq [-n+i, 0]$;

(e) $\forall i < j < n, f(W_{i,j}) \subseteq \{0\}$ if

$$h_{L(i)}(y_j) \notin \{y_k : k \in n\} \cup \{h_{L(j)}(y_j)\},$$

otherwise $f(W_{i,j}) \subseteq [-n+j, n]$;

(f) $f(X \backslash [\bigcup \{v_i : i < n\} \cup \bigcup \{W_{i,j} : i \leq j < n\}]) \subseteq \{0\}$.

To construct such a function, $f$, first for $i\in n$ choose functions $g_i:X\to[0,i+1]$ with $g_i(y_i)=i+1$ and $g_i(X\setminus V_i)=\{0\}$. Next for $i\in n$ choose functions $g_{n+i}:X\to[-n+i,0]$ with $g_{n+i}(h_{L(i)}(y_i))=-n+i$ and $g_{n+i}(X\setminus W_{i,i})=\{0\}$. Now by our choice of $V_i$'s and $W_{i,i}$'s we have that the nonzero portions of these functions are all pairwise disjoint. (See conditions (i) and (iii) above.) So if we add all of these we get a suitable function

$$f(x)=\sum\{g_i(x)\colon i\in 2n\}.$$

Where for condition (e) recall that $\forall i<j<n$,

$$h_{L(i)}(y_j)\notin\{h_{L(k)}(y_k)\colon k\in n\setminus\{j\}\}.$$

Let $F=H_L(f)$ $(\in C_P(X^n))$. $F$ will separate $y$ and the closed set $A$. Note that by definition of $H_L$ and conditions (a) and (c) $F(y)=0$. We will finish the proof of Claim 2 by showing that $F^{\leftarrow}[0,\frac{1}{2})$ is contained in $\prod_{i\in n} V_i$.

Let $z=\langle z_i : i\in n\rangle\in F^{\leftarrow}[0,\frac{1}{2})$. Then $\forall i\in n$,

$$|f(z_i)-(i+1)|<\tfrac{1}{2}.$$

So by (b), (d), and (f) we have

$$\forall i\in n,\quad z_i\in\bigcup\{V_j\colon i\leq j<n\}.$$

Assume that $z_i\notin V_i$ for some $i\in n$. Then, since the $V_j$'s are pairwise disjoint and $n$ is finite, $\exists i<j\leq k$ such that

$$\{z_i, z_j\}\subseteq V_k.$$

But then $h_{L(i)}(z_i)\subseteq W_{i,k}$ by our choice of $V_k$ (see (v)). Hence $f(h_{L(i)}(z_i))\in[-n+k,n]$ by (e). But $i<k<n$ implies that the quantity $-n+i+\frac{1}{2}<-n+k$. But we also have that

$$|f(h_{L(i)}(z_i))+n-i|<\tfrac{1}{2}.$$

So $f(h_{L(i)}(z_i))<-n+\frac{1}{2}+i<-n+k$. This yields a contradiction to the assumption that $z_i\notin V_i$ for some $i\in n$. Hence $z\in\prod V_i$ as we desired. $\square$

**Notation 4.7.** Fix the following notation for the rest of the paper. For a collection of continuous functions $\{h_i : i\in I\}$ and for $i,j\in n$ $(i\neq j)$ and $r\in I$ define

$$A^r_{i,j}=\{\langle x_k : k\in n\rangle\in X^n\colon x_i=h_r(x_j)\}.$$

For a collection of homeomorphic embeddings $\{h_i : i\in I\}$ and for $i,j\in n$ $(i\neq j)$ and $r,s\in I$ define

$$B^{r,s}_{i,j}=\{\langle x_k : k\in n\rangle\in X^n\colon x_i\in h_s^{\leftarrow}(\mathrm{Dom}\ h_r^{-1})\wedge x_j=h_r^{-1}(h_s(x_i))\}.$$

Then the $A^r_{i,j}$'s are continuous images of $X^{n-1}$ and the $B^{r,s}_{i,j}$'s are continuous images of $X^{n-2}\times h_s^{\leftarrow}(\mathrm{Dom}\ h_r^{-1})\subseteq X^{n-1}$.

**Proof of Theorem 4.5.** We finish by showing that $X^n$ is covered by the union of the $Y_L$'s, the $\Delta_{i,j}$'s, the $A^r_{i,j}$'s, and the $B^{r,s}_{i,j}$'s.

Let $x = \langle x_i : i \in n \rangle \in X^n \backslash \bigcup \Delta_{i,j}$ and choose an $L \in {}^nI$ such that

$$\forall k \in n, \quad h_{L(k)}(x_k) \neq x_k. \tag{$*$}$$

Now if $\exists i, j \in n$ such that $h_{L(j)}(x_j) = x_i$ then $x \in A^{L(j)}_{i,j}$ where $i \neq j$ by $(*)$. So without loss of generality assume that $x \in X^n \backslash [\bigcup \Delta_{i,j} \cup \bigcup A^r_{i,j}]$.

*Case* 1. $\exists i < j < n$ such that $h_{L(i)}(x_i) = h_{L(j)}(x_j)$. But this gives $x_i = h^{-1}_{L(i)}(h_{L(j)}(x_j))$. So $x \in B^{L(i),L(j)}_{j,i}$.

*Case* 2. $\exists i < j < n$ and $k \in n \backslash \{j\}$ such that $h_{L(i)}(x_j) = h_{L(k)}(x_k)$. In this case $x_j = h^{-1}_{L(i)}(h_{L(k)}(x_k))$. So $x \in B^{L(i),L(k)}_{k,j}$.

Finally if none of the above hold then $x \in Y_L$. □

**Theorem 4.8.** *If $A \subseteq X$ and there is a continuous,* 1-1 *function $f : A \to X$ with $f(A) \cap A = \emptyset$ (in particular a copy of $A \times 2 \subseteq X$), then* $\mathrm{hd}^*(A) \leqslant \mathrm{hL}[C_P(X)]$ *and* $s^*(A) \leqslant s[C_P(X)]$.

**Proof.** Fix $A$ and $f$ as in the theorem. We will consider the map $H_L$ in Lemma 4.6 to be from $C_P(X)$ to $C_P(A^n)$ where $L$ is just a constant function (there is only one "$h_i$", namely the function $f$). Assume the theorem is false. Then let $n \in \mathbb{N}$ be the least $n$ with a discrete counter example. Fix a discrete $S \subseteq A^n$,

$$S = \{\langle s_{i,\alpha} : i \in n \rangle : \alpha \in \kappa^+\}.$$

Without loss of generality if $\langle s_{i,\alpha} : i \in n \rangle$ and $\langle s_{j,\beta} : j \in n \rangle$ are in $S \subseteq A^n$, and $\alpha \neq \beta$ or $i \neq j$ we have $s_{i,\alpha} \neq s_{j,\beta}$. But then $S \subseteq Y_L$ (as in Lemma 4.6) by the hypothesis on $f$. We then get a contradiction using Lemma 4.6. □

**Theorem 4.9.** *Suppose for all infinite $A \subseteq X$ there is a continuous,* 1-1 *function $f_A : A \to X$ with $|\{a \in A : f(a) = a\}| < |A|$. Then $s^*(X) = s[C_P(X)]$ and* $\mathrm{hd}^*(X) = \mathrm{hL}[C_P(X)]$.

**Proof.** We modify the above proof to show that if $S$ is a discrete subset in $X^n$, then there is an $S' \subseteq S$ with the same cardinality which is covered by a $Y_L$ (as defined in Lemma 4.6) and the $A^r_{i,j}$'s.

Assuming the theorem is true for $n-1$. Suppose $S \subseteq X^n$ is a discrete counter example. First, without loss of generality, $S \cap (\bigcup \Delta_{i,j}) = \emptyset$ by the induction hypothesis. Second, without loss of generality, $S = \{\langle s_{i,\alpha} : i \in n \rangle : \alpha \in \kappa^+\}$ where $s_{i,\alpha} = s_{j,\beta}$ iff $i = j$ and $\alpha = \beta$. Because if any coordinate is repeated consider $S \cap (X^{n-1} \times \{s_{i,\alpha}\})$. This must have cardinality at most $\kappa$ by the induction hypothesis so we may choose a subset of $S$ with the desired property.

Let $A = \{s_{i,\alpha} : \langle i, \alpha \rangle \in n \times \kappa^+\}$ and $f$ the function guaranteed by the theorem hypothesis. Now consider

$$S' = \{\langle s_{i,\alpha} : i \in n \rangle : \forall i \in n \; f(s_{i,\alpha}) \neq s_{i,\alpha}\}.$$

$S'$ still has cardinality $\kappa^+$ since we removed at most the $\kappa$ elements. However $S'$ can be covered by a $Y_L$ (where $L$ is a constant function) and the $A^r_{i,j}$'s. But this gives us a contradiction since each of this finite list of sets has spread at most $\kappa$. The equality for $\mathrm{hd}^*(X)$ and $\mathrm{hL}[C_\mathrm{P}(X)]$ holds by Theorem 3.5. □

Finally we show how to apply Theorem 4.5 to Lindelöf homogeneous spaces.

**Lemma 4.10.** *If $X$ is a Lindelöf homogeneous space, then $X$ has a countable collection of homeomorphisms, $h_i$, such that $\forall x \in X$ $\exists h_i$ with $h_i(x) \neq x$.*

**Proof.** For $\langle x, y\rangle \in X^2 \backslash \Delta$ fix a homeomorphism $f_{x,y}: X \to X$ with $f_{x,y}(x) = y$. Let $O_{x,y} = \{x \in X: f_{x,y}(z) \neq z\}$. Then $O_{x,y}$ is an open set containing $x$. Hence there is a countable set $J \subseteq X^2 \backslash \Delta$ such that $X \subseteq \bigcup \{O_{x,y}: \langle x, y\rangle \in J\}$. Hence the collection of homeomorphisms $\{f_{x,y}: \langle x, y\rangle \in J\}$ will work. □

**Theorem 4.11.** *Lindelöf homogeneous spaces $X$, satisfy the equalities* $\mathrm{hd}^*(X) = \mathrm{hL}[C_\mathrm{P}(X)]$ *and* $s^*(X) = s[C_\mathrm{P}(X)]$. *(In particular this is true for the compact case.)*

**Proof.** Apply Lemma 4.10 and Theorem 4.5. □

## References


[1] A.V. Arhangel'skii, A survey of $C_\mathrm{P}$-theory, Questions Answers Gen. Topology 5 (1987) 1-109.

[2] A.V. Arhangel'skii, On the hereditary-Lindelöfness of the space of continuous functions, Vestnik Moskov. Univ. Ser. I Mat. Mekh. 3 (1989) 67-69.

[3] M.O. Asonov, About the space of continuous functions, in: A. Császár, ed., Topology Theory and Applications, Colloquia Mathematica Societatis János Bolyai 41 (North-Holland, Amsterdam, 1983) 31-34.

[4] G. Gruenhage, Games, covering properties and Eberlein compacts, Topology Appl. 23 (1986) 291-297.

[5] R. Hodel, Cardinal functions I, in: Handbook of Set-Theoretic Topology (North-Holland, Amsterdam, 1984) 1-62.

[6] J. Juhasz, Cardinal functions II, in: Handbook of Set-Theoretic Topology (North-Holland, Amsterdam, 1984) 63-109.

[7] N. Noble, The density character of function spaces, Proc. Amer. Math. Soc. 42 (1) (1974) 228-233.

[8] N.V. Veličko, Weak topology of spaces of continuous functions, Mat. Zametki 30 (5) (1981) 703-712.

[9] P. Zenor, Hereditary $m$-separability and the hereditary $m$-Lindelöf property in product spaces and function spaces, Fund. Math. 106 (1980) 175-180.